

Scheduling Algorithms for Procrastinators

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Abstract

This paper presents scheduling algorithms for procrastinators, where the speed that a procrastinator executes a job increases as the due date approaches. We give optimal off-line scheduling policies for linearly increasing speed functions. We then explain the computational/numerical issues involved in implementing this policy. We next explore the online setting, showing that there exist adversaries that force any online scheduling policy to miss due dates. This impossibility result motivates the problem of minimizing the *maximum interval stretch* of any job; the interval stretch of a job is the job's flow time divided by the job's due date minus release time. We show that several common scheduling strategies, including the "hit-the-highest-nail" strategy beloved by procrastinators, have arbitrarily large maximum interval stretch. Then we give the "thrashing" scheduling policy and show that it is a $\Theta(1)$ approximation algorithm for the maximum interval stretch.

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1 Introduction

We are writing this sentence two days before the deadline. Unfortunately that sentence (and this one) are among the first that we have written. How could we have delayed so much when we have known about this deadline for months? The purpose of this paper is to explain why we have waited until the last moment to write this paper.

In our explanation we model procrastination as a scheduling problem. We cannot use traditional scheduling algorithms to model our behavior because such algorithms do not take into account our (and humanity's) tendency to procrastinate. The advantages of procrastination are well documented: the closer to a deadline a task is executed, the less processing time the task appears to require. Hence, it is common for a person to delay executing some onerous job in order to spend as little time as possible working on it.

Regarding this paper, it will certainly be written quickly — it will have to be, since the deadline is near. Perhaps we will write faster under pressure because we will expend less time overanalyzing each design option. Other aspects of the paper may change because of this time pressure. In any case, the writing will proceed faster than if we had begun earlier.

Our scheduling problem for procrastinators is unorthodox in that the processing time of a job depends on the times during which the job is run. We are given as input a set of jobs $\mathcal{J} = \{1, 2, \dots, n\}$. Each job j has release time r_j , due date d_j , and work w_j ; without loss of generality, we assume that the jobs are indexed by increasing release times. *Preemption* is allowed; that is, a running job can be interrupted and resumed later. The speed at which job j is run depends on the times that j is executed; the closer to the due date d_j , the faster j can be executed. Specifically, *speed function* $f_j(t)$ indicates that at time t , job j is executed with speed $f_j(t)$; thus, if j is executed during time interval $[a, b]$, then $\int_{t=a}^b dt f_j(t)$ units of work of job j complete.

Throughout most of the paper we focus on *linear* speed functions. We assume that when job j first is released, it is executed with speed 0. In accordance with this last assumption, when the call for papers first appeared, we snapped into action and accomplished nothing.

Despite our whimsical and self-referential style, we hope to emphasize that the scheduling problems on streams with time-dependent processing times have mathematical subtlety as well as practical relevance. The time-dependent processing models in this paper may be useful for industry and sociology because they give better scheduling models of human behavior; no model can truly be accurate that does not account for people's ability to work faster under the temporary stress of deadlines. More generally, many common scheduling problems in both daily life and industry have tasks whose processing times are time-dependent. For example, an airplane that is late in arriving may have the boarding procedure expedited, a construction project that is behind may have more workers assigned to it, and a shipment that is late may be delivered faster. Indeed a major reason for the success of companies such as Fedex, UPS, and DHL is that the world is filled with scheduling problems executed by procrastinators.

Related Work

A number of other optimization problems have well studied time-dependent variants, including work on time-dependent shortest paths [24] and time-dependent flows [16, 15]. Some authors, typically in the operations-research community, have also worked on scheduling with time-dependent processing times (see, e.g., [2, 5, 17, 18]), but for the offline and nonpreemptive case. Of course, preemptive and online models are best for modeling the behavior of procrastinators, who tend to timeshare and thrash as the deadlines approach. Moreover, our introduction of preemptive scheduling with time-dependent processing times requires an entirely different model. Previous work has assumed that the processing time $p_j(t)$ for job j is a function of the starting time t . We cannot have such a model in a preemptive case because the job may be executed during many different time intervals. This issue motivates our need for processor speeds: job

j is executed with speed $f_j(t)$ at time t ; the processing time is the sum over all intervals during which job j is executed, and the integral of $f_j(t)$ over all times that the job is executed must equal the job’s work. Curiously, if we analyze existing nonpreemptive models (e.g., linearly decreasing processing times) and analyze what processor speeds and total work must be to generate these processing times, then we can create instances where the processing speeds at times approach infinity; clearly such a model is unrealistic.

The most closely related work in the literature is on scheduling algorithms for minimizing power consumption and, in particular, on “speed scaling.” See [6, 25, 10, 7, 26, 1] for some recent results and [20] for an excellent survey. The idea of speed scaling is that the processing speed of a job is variable, but faster speeds consume more power. This ability to vary the speeds is reminiscent of the procrastinator who can run at unsustainable rates near the deadline. However, unlike in the speed-scaling model, the procrastinator has less freedom in choosing the processing speed; the processing speed is solely determined by the proximity to the deadline.

We note that there exist other scheduling papers where processors have different speeds, both for “related” processors [12, 11, 9] and for “unrelated” processors [13, 23, 21]. However, neither situation models procrastination scheduling (or speed scaling), where the processing speeds per job change over time.

There are other scheduling problems on how to schedule reluctant workers, such as the lazy bureaucrat problem [3, 19, 4]. However, the lazy bureaucrats in the scheduling problem are trying to accomplish as few of the jobs as possible, whereas the procrastinators in the current scheduling problem are trying to finish all of the jobs.

Results

In this paper we present the following results.

- *Optimal offline scheduling* — We first give *optimal* offline scheduling policies for the case where a scheduling instance has a feasible solution. We consider the case of linear speed functions, $f_j(t) = m_j(t - r_j)$, for constant $m_j \geq 0$. (In the offline problem, the scheduler sees the entire problem instance before it has to begin scheduling.) Specifically, the policy gives the feasible solution in which the processors spend the minimum total time running. These results are consistent with a procrastinator who, after missing crucial deadlines, muses “if I could do it all over again. . .”
- *Computational/numerical issues* — We show that, curiously, despite a simple optimal scheduling policy, actually *determining feasibility* of the resulting schedule is not even known to be in NP. In particular, determining feasibility is hard because of the computational difficulties of summing square roots. We know of few scheduling problems where this intriguing issue arises.
- *Online scheduling* — We next turn to online scheduling. Not surprisingly, the feasibility problem is not achievable in an online setting. In particular, even if the online procrastinator has a feasible set of jobs, he/she may be forced to miss an arbitrarily large number of due dates.
- *Online maximum interval stretch* — A procrastinator may be forced to execute jobs beyond their due dates, that is, for some job j , the completion time C_j may exceed the due date d_j . Generally speaking, if a procrastinator has a year to do a job j , and completes j two weeks late, the situation is better than if the procrastinator has only one day to do j , but completes two weeks late. This observation motivates the notion of *interval stretch*, defined as the flow time (time the job spends in the system) divided by the job’s interval. More formally, the interval stretch¹ of job j is defined as $s_j = (C_j - r_j)/(d_j - r_j)$.

¹This definition deviates from the standard notion of stretch where the flow time is divided by the total time the job has spent working [8]. However, it is appropriate here as jobs have due dates which can be missed and job speed is time-dependent.

We consider the optimization metric *maximum interval stretch* (abbreviated to *max-stretch*), to be defined as $\max_j s_j$.

We first explore traditional scheduling policies for the procrastinator, such as First-In-First-Out (FIFO), Shortest-Remaining-Processing-Time (SRPT), and earliest-due-date (EDD). We show, not surprisingly, that these policies do not perform well and can lead to unbounded max-stretch. A common scheduling policy among many procrastinators is “hit-the-highest-nail”, that is, execute the task that most crucially requires attention, formally, Largest-Stretch-So-Far (LSSF). In LSSF we execute the job in the system that *currently* has the largest interval stretch. We prove, perhaps surprisingly, that LSSF can lead to arbitrarily large max-stretch. We conclude our exploration of max-stretch by exhibiting an online algorithm for the procrastinator, THRASHING, that yields $\Theta(1)$ max-stretch even when the maximum job speed is bounded.

2 Offline Procrastination Scheduling

In this section we consider the *offline* procrastination-scheduling problem. First, we give an optimal scheduling policy based on a simple priority rule. Then we show that it is computationally difficult to determine whether a scheduling instance is feasible, despite this priority rule. We focus on linear speed functions, $f_j(t) = m_j(t - r_j)$.

Optimal Offline Scheduling Policy

We now give an optimal scheduling policy for the offline procrastination problem based on a simple priority rule.

We first define terms. We say that a schedule is *feasible* if all jobs complete within their intervals; we say that a feasible schedule is *optimal* if the total processing time is minimized. Observe that if an optimal schedule has no idle time then all feasible schedules are also optimal.

The optimal algorithm starts at the latest due date and works backwards in time, prioritizing jobs by the latest release time. Whenever a new job is encountered (at the job’s due date) or a job completes, then the job in the system having the latest release time is serviced. Where two or more jobs have the same release time the scheduler chooses between them in an arbitrary but fixed way. We call this scheduling algorithm *Latest Release Time Backwards (LRTB)*.

Observe that LRTB is the traditional Earliest Due Date (EDD) policy (see, e.g., [22]) when we reverse the flow of time so that release dates become due dates and due dates become release dates. In traditional scheduling, time can flow in either direction, so that both LRTB and EDD generate feasible schedules. In contrast, in the procrastination problem, EDD performs poorly; see Section 3. Observe that the job priorities depend only on the release times and not the slopes. This lack of dependence on the slopes should not be surprising because we can transform any scheduling instance into an instance having all unit slopes by setting the workload to be $w'_i = w_i/m_i$. The intuition of the algorithm is that it always tries to push the work of a job as near to its due date as possible in order to maximize the processing speed.

In the following we prove that algorithm LRTB produces the optimal schedule.

Theorem 1 *LRTB is an optimal algorithm for the procrastination scheduling problem.*

Proof. The proof is by an exchange argument. We first assume that no two jobs have the same release time and then relax that assumption at the end. Suppose for the sake of contradiction that there exists an optimal schedule A different from LRTB. Specifically, these schedules differ in the order of execution of two jobs with different release times. We perform a single exchange of work to yield another feasible schedule A^* having smaller total processing time than A , thus obtaining a contradiction.

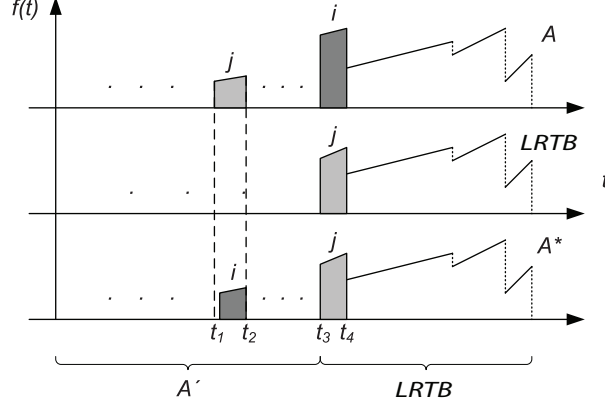


Figure 1: Schedule A^* results from the merge of A' and LRTB. Schedule A' results from A by exchanging jobs j and i . The small gap after t_1 indicates that this exchange is more time efficient.

We now define terms. Consider the latest instant in time where LRTB differs from A and call this time t_4 . Consider an arbitrarily small interval $[t_3, t_4]$, during which job j runs in LRTB and job i runs in A . See Figure 1 for a depiction of the setting. By the definition of LRTB, i , j , and t_4 , $r_i < r_j$. Consider some earlier time interval $[t_1, t_2]$, i.e., $t_2 \leq t_3$, during which job j runs in A . Define t_1 , t_2 , and t_3 so that the amount of work that can be executed on job j is the same, that is,

$$\int_{t=t_1}^{t_2} dt f_j(t) = \int_{t=t_3}^{t_4} dt f_j(t).$$

Now we make a new schedule A^* from A by exchanging the work done during intervals $[t_1, t_2]$ to $[t_3, t_4]$. Specifically in A^* , job j is run during $[t_3, t_4]$ and job i is run during $[t_1, t_2]$. We know that this exchange is allowed because $d_i > t_4$ and $d_j > t_4$ (from the LRTB and A schedules) and because $r_i < r_j \leq t_1$ (from the A schedule and because $r_i < r_j$). By the definition of the intervals, exactly the same amount of work on j can be done during both intervals. Computing the area of the trapezoids defined by the $f_j(t)$ we obtain

$$(t_4 - t_3) \left(\frac{t_4 + t_3}{2} - r_j \right) m_j = (t_2 - t_1) \left(\frac{t_2 + t_1}{2} - r_j \right) m_j,$$

meaning that

$$(t_4^2 - t_3^2)/2 - r_j(t_4 - t_3) = (t_2^2 - t_1^2)/2 - r_j(t_2 - t_1). \quad (1)$$

Observe that $t_4 - t_3 < t_2 - t_1$ because the speed that j is executed during $(t_3, t_4]$ is greater than during $[t_1, t_2]$.

The amount of work on job i that needs to be exchanged from $[t_3, t_4]$ to $[t_1, t_2]$ is $(t_4^2 - t_3^2)/2 - (t_4 - t_3)r_i$. But since $r_i < r_j$ and $t_4 - t_3 < t_2 - t_1$,

$$(t_4^2 - t_3^2)/2 - r_i(t_4 - t_3) < (t_2^2 - t_1^2)/2 - r_i(t_2 - t_1),$$

and therefore interval $[t_1, t_2]$ is big enough to execute all of the work on job i and still leave some idle time. Hence, schedule A^* is feasible and spends a smaller amount of time working. This gives us our contradiction.

We now explain the case where two jobs 1 and 2 have the same release time. Assume that job 1 is scheduled to execute some work in the time interval $[t_1, t_2]$ and job 2 is scheduled to execute some work in the interval $[t_3, t_4]$. If we exchange the work for jobs 1 and 2, the relationship between the new time intervals and the old is expressed by the simple equation $t_4^2 - t_3^2 = t_2^2 - t_1^2$. Therefore the total time to execute both

jobs remains the same after exchange. As a result, the order in which these jobs are executed does not affect the total processing time, and so LRTB is an optimal algorithm no matter what the tie-breaking rule is. This completes the proof. \square

Determining Feasibility may not be in NP

One of the remarkable features of the procrastination problem is that, despite having the simple optimal scheduling policy LRTB, it is unclear whether determining the feasibility of a scheduling instance is even in NP, even for linear speed functions.

The difficulty is numerical. Calculating the actual processing time of the job j given a starting or ending time t and speed function $f_j(t) = t - r_j$ requires computing square roots. Determining the feasibility of the schedule therefore requires computing sums of square roots and their relationship to an integer, and this problem appears to be numerically difficult.

The basic sum-of-square-roots problem is to determine whether

$$\sum_{i=1}^n \sqrt{x_i} < I$$

for some $x_i, I \in \mathbb{Z}$ ($1 \leq i \leq n$). Because this problem is not known to be in NP, basic computational-geometry problems such as Euclidean TSP or Euclidean shortest paths are not known to be in NP. See the Open Problems Project [14, Problem 33] for a nice discussion of the sum-of-square-roots problem.

We establish the difficulty of procrastination scheduling by providing a reduction from any instance of the sum-of-square-roots problem. To derive the cleanest reduction, we allow the existence of non-lazy jobs, i.e., jobs that are always executed at the same speed, i.e., having slope 0. (It is likely that a reduction can be made to work using no non-lazy jobs, but at the cost of additional complications.)

Theorem 2 *The procrastination scheduling is not in NP unless the sum-of-square-roots problem is also in NP.*

Proof. We reduce the sum-of-square-roots problem to the procrastination scheduling problem. Given numbers x_1, \dots, x_m and integer I , we will create a procrastination-scheduling problem with $m + 1$ jobs. The procrastination scheduling problem will be feasible if and only if $\sum_{i=1}^m \sqrt{x_i} \geq I$.

We first give the structure of the scheduling instance and then determine the release times, deadlines, and work for each job. In our scheduling instance, the first m lazy jobs $1 \dots m$ have non-overlapping intervals, so that $r_1 = 0$, and the due date of one job is the release date of the next: $r_{i+1} = d_i$ ($i = 1, \dots, m - 1$). The $(m + 1)$ st job will be nonlazy. We place this job's interval so that it overlaps with the intervals of jobs $1, \dots, m$, i.e., $r_{m+1} = r_1$ and $d_{m+1} = d_m$.

We now construct the lazy jobs $1, \dots, m$. For job i , we choose interval length ℓ_i ($= d_i - r_i$) and work w_i to be positive integers such that $\ell_i^2 - 2w_i = x_i$; many choices of ℓ_i and w_i will work. It will suffice to choose positive integers ℓ_i and w_i such that $0 < \ell_i^2 - 2w_i < \ell_i$. Lazy job i runs most efficiently when pushed to the right side of its interval. Then it runs in time $\ell_i - \sqrt{\ell_i^2 - 2w_i}$. The total time taken by all m non-overlapping jobs when scheduled optimally is therefore

$$\sum_{i=1}^m \ell_i - \sum_{i=1}^m \sqrt{\ell_i^2 - 2w_i} = \sum_{i=1}^m \ell_i - \sum_{i=1}^m \sqrt{x_i}.$$

We now construct the nonlazy job $m + 1$. As described earlier $r_{m+1} = 0$ and $d_{m+1} = d_m$. We set the work $w_{m+1} = I + d_{m+1} - \sum_{i=1}^m \ell_i$.

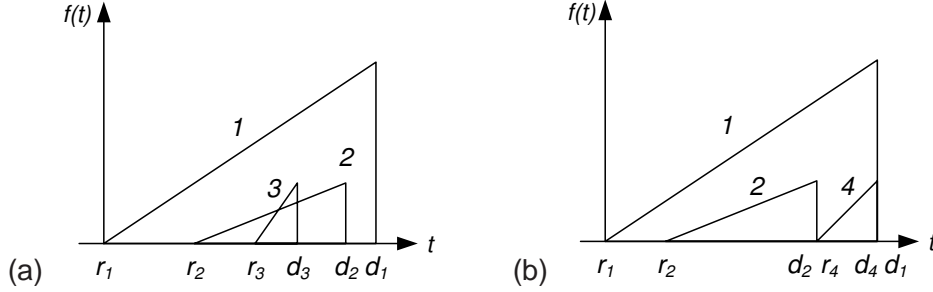


Figure 2: (a) Case 1: job j_1 is run at r_2 . Then job 3 arrives. Either job 2 or 3 is forced to miss its due date. (b) Case 2: job j_2 is run at r_2 . Then job 4 arrives. Either job 1, 2, or 4 is forced to miss its due date.

Now there is a feasible solution for this problem iff $d_{m+1} \geq w_{m+1} + \sum_{i=1}^m \ell_i - \sum_{i=1}^m \sqrt{x_i}$. This is the case, as long as $I \leq \sum_{i=1}^n \sqrt{x_i}$. Thus, an arbitrary instance of the sum-of-square-roots problem can be reduced to an instance of procrastination scheduling, implying the numerical difficulty of procrastination scheduling. \square

3 Online Algorithms

This section considers the online procrastination scheduling problem. In the online problem, jobs $1 \dots n$ arrive over time. Job j is known to the scheduler only at the release time r_j , at which point the scheduler also learns the values of w_j and d_j . We first show that it is difficult for an online scheduler to find feasible schedules. Next we search for online algorithms that generate small, ideally constant, max-stretch. We then examine traditional scheduling policies such as EDD, SRPT, and FIFO, and we show that these policies have large, typically unbounded, max-stretch. We next consider the scheduling policy Largest-Stretch-So-Far (LSSF), which executes the job in the system currently having the largest interval stretch. This policy formalizes the “hit-the-highest-nail” scheduling policy, that is, execute the task in the system that most crucially requires attention. More precisely, in the LSSF scheduling policy, we run the job in the system that has incurred the largest interval stretch so far, that is, at time t we execute the job j that maximizes $(t - r_j)/(d_j - r_j)$. We show that, remarkably, LSSF also has unbounded max-stretch. We conclude this section by exhibiting the scheduling algorithm THRASHING, whose max-stretch is within a constant factor of optimal and then give a generalization to non-linear speed functions. It seems unrealistic in our procrastination model to assume that the procrastinator can work arbitrarily fast. One consequence of this last result is that good online max-interval-stretch bounds are achievable even when the procrastinator’s maximum processing speed is at most a constant factor faster than a nonprocrastinator’s speed.

Basic Results

We first show that any online algorithm can be forced to miss due dates, even when the scheduling instance is feasible. A job j has *slack* if the work, w_j , associated with it is less than the area between r_j and d_j .

Theorem 3 *For any online algorithm, there is a feasible job stream on which that algorithm misses due dates.*

Proof. We show that regardless of the online scheduling decisions, the adversary can force the algorithm to miss due dates by maliciously selecting future jobs. The adversary first sends jobs 1 and 2, where $r_1 < r_2$ and $d_2 < d_1$. Both jobs 1 and 2 have some slack and the set $\{1, 2\}$ is feasible. At time r_2 there are two cases:

1. Job 1 is serviced at time r_2 . Then the adversary places a job 3 with $r_2 < r_3 < d_3 < d_2$. Job 3 is designed so that the entire interval $[r_2, d_2]$ is required to complete jobs 2 and 3 by their due dates. Since the online algorithm works partially on job 1 during this interval, either job 2 or 3 misses its due date; see Figure 2(a).
2. Job 2 is serviced at time r_2 . The adversary places a job 4 with $r_4 > d_2$ and $d_4 < d_1$. Job 4 is designed so that all the time between r_2 and d_1 is required to complete jobs 1, 2, and 4 by their due dates. However, as job 2 has some slack we know that by Theorem 1 that the optimal strategy is to run 1 at time r_2 and that this strategy is unique. Therefore, by running 2 at time r_2 the algorithm misses at least one of the due dates; see Figure 2(b). \square

Note that by repeating this construction, the adversary can force the algorithm to miss an arbitrarily large number of due dates. Thus, Theorem 3 explains why procrastinators may have a harder time juggling online tasks than non-procrastinators.

We now show that most traditional scheduling policies for non-procrastinators do not work well for procrastinators. The following theorem gives the performance of First-In-First-Out (FIFO), Earliest-Due-Date (EDD), and Shortest-Remaining-Processing-Time (SRPT).

Theorem 4 *The max-stretch of the First-In-First-Out (FIFO) and Earliest-Due-Date (EDD) scheduling policies can be made arbitrarily large, even for a constant number of jobs. For a set of n jobs, the Shortest-Remaining-Processing-Time (SRPT) scheduling policy can have a max-stretch of $\Omega(\sqrt{n})$.*

Hitting the Highest Nail Does Not Work

A common scheduling strategy among procrastinators is “Hit the Highest Nail,” that is, execute the job that is farthest behind. Since the objective is to minimize the max-stretch, “hitting-the-highest-nail” translates to running the job that has the largest interval stretch. We call this strategy *Largest-Stretch-So-Far (LSSF)*. More precisely, in the LSSF scheduling policy, we run the job in the system that has incurred the largest interval stretch so far, that is, at time t we execute the job j that maximizes $(t - r_j)/(d_j - r_j)$. Thus, the algorithm might execute a job i , but switch to a smaller job j that arrived after i , once j 's interval stretch so far surpasses that of i 's.

Remarkably, LSSF also leads to jobs having unbounded max interval stretch. The overall strategy of our malicious adversary is as follows. First arrange three jobs so that one of them is not started by LSSF until it reaches its due date. This can be achieved by choosing jobs 1 and 2 so that $r_2 > r_1$ and $d_2 < d_1$. LSSF schedules 1 to work uninterrupted until some point in the interval of 2 at which point it starts work on 2. Assuming 2 has no slack, it complete its work after its due date. We can now place job 3 so that $r_3 = d_2$ and set d_3 to be the completion time of job 2. Job 3 will not start until its due date, d_3 , and assuming it works uninterrupted and has no slack, completes with an interval stretch of $\sqrt{2}$. See Figure 3(a).

Let x be the finishing time of job 3. Our malicious adversary now schedules a stream of jobs with no slack starting at d_3 , as follows. See Figure 3(b).

1. Place a job j with release date d_3 so that its stretch is exactly $\sqrt{2}$ at x . This requires setting the length of the job appropriately. This new job does not start work until x and so finishes at some further point x' . The stretch of j is now greater than $\sqrt{2}$. Call this new stretch value α .
2. Place a second job j' starting at the due date of j so that its stretch is exactly α at time x' . j' has the due date of j as its release date. Now j' does not start work until x' and finishes at some later date x'' with an extended stretch of $\alpha' > \alpha$.
3. Iterate.

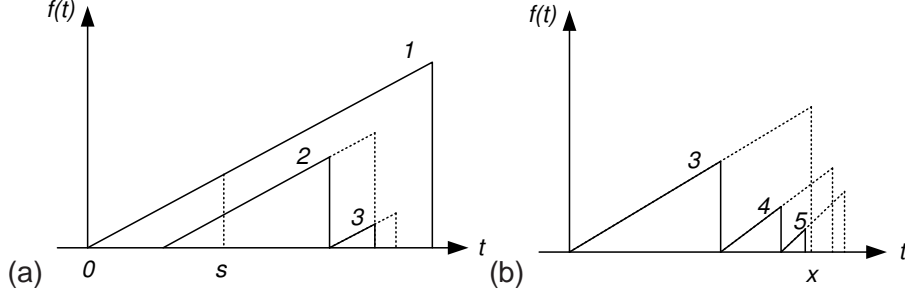


Figure 3: (a) Job 2 starts work at s and completes after its due date. Job 3 starts work at its due date and finishes with a stretch of $\sqrt{2}$ (b) A stream of jobs with increasing stretch. The stretch of job 4 is $\sqrt{2}$ when it starts work at time x and $\sqrt{3}$ when it finishes.

Theorem 5 Using the LSSF scheduling policy, the max-stretch for a feasible set of n jobs is $\Omega(\sqrt{n})$.

Proof. We analyze the LSSF scheduling policy by considering in turn each job that the malicious adversary places using the procedure above and we calculate the update rule for the max-stretch. Because we only consider one job at a time, we can assume that the release date is 0. We use three variables, the current stretch α , the time ℓ at which the current job can start work, and the time x at which the current job finishes working. The due date for the current job is therefore ℓ/α for jobs $j \geq 4$. By equating areas we have

$$x^2 - \ell^2 = \frac{\ell^2}{\alpha^2} \Rightarrow x^2 = \frac{\ell^2}{\alpha^2} + \ell^2$$

The update rule for stretch is

$$\alpha' = \frac{x\alpha}{\ell} \Rightarrow \alpha'^2 = \frac{x^2\alpha^2}{\ell^2}.$$

Therefore

$$\alpha' = \sqrt{1 + \alpha^2}.$$

Recasting as the recurrence relation $\alpha[n] = \sqrt{1 + \alpha[n-1]^2}$ and setting the boundary condition to be $\alpha[1] = 1$ we have

$$\alpha[n] = \sqrt{n}.$$

Therefore, the max-stretch is $\Omega(\sqrt{n})$ as required.

It remains to be proven that a feasible schedule for the arrangement of jobs described exists. In particular, as job 1 is the only job with any slack, we must show that it is possible to complete 1 by its due date without performing any of its work in the interval of another job. It is sufficient to prove that the amount of time taken by all the other jobs is bounded above by a constant. The remaining work of job 1 can then be completed in the interval between the due date of job n and the due date of job 1.

Following the same line of reasoning as above, the update rule for ℓ is $\ell' = x - \ell/\alpha$. Therefore

$$\frac{\ell'}{\alpha'} = \frac{\ell}{\alpha} \left(1 - \frac{1}{\sqrt{1 + \alpha^2}} \right)$$

gives us the length of the next job inserted into the schedule. Assume, w.l.o.g. that the length of job 3 is 1. The size of job $n \geq 4$ is therefore $\prod_{i=1}^{n-3} (1 - 1/\sqrt{i+1})$ which is $\Theta(e^{-\sqrt{n}})$. The total time taken by all the jobs $4 \dots n$ is therefore $\Theta(\sum_{i=1}^n e^{-\sqrt{i}})$, which is $\Theta(1)$.

By setting the sizes of jobs 2 and 3 to be constant, we have the required constant bound for the total size of all jobs $j > 1$. \square

$\Theta(1)$ -Competitive Online Algorithm for Max-Stretch

We now exhibit the strategy THRASHING, which bounds the interval stretch of each job by 4. The THRASHING strategy models the extreme case of a procrastinator who does not work on any job until it has already passed its due date. More formally, in this strategy no job is executed until it has a stretch of at least 2. Among all such jobs, the procrastinator executes the job that arrived latest.

We begin by proving the following simple lemma:

Lemma 6 *Consider a feasible set of jobs $1, \dots, m$ and consider times r and d , where all $r_j \geq r$ and $d_j \leq d$. Let α -DLY be any scheduling policy that only schedules work from jobs having stretch at least α , where $\alpha \geq 1$. The total amount of time required to run all jobs using α -DLY is at most $(d - r)/\alpha$.*

Proof. Because the set of jobs is feasible, there is some way to schedule each job within its interval and the total time spent working is at most $d - r$. Now consider running α -DLY. For any given job j , the slowest that j runs in α -DLY is at least α times faster than j runs in the feasible schedule. The lemma follows immediately. \square

Theorem 7 *For any feasible set of jobs, THRASHING bounds the interval stretch of every job by 4.*

Proof. The proof is by contradiction. Define the *extended due date* \tilde{d}_j of job j to be the time that j must complete by to guarantee an interval stretch of 4, that is, $\tilde{d}_j = 4(d_j - r_j) + r_j$. Consider some job j that does not meet its extended due date. For simplicity and without loss of generality, we normalize time so that $r_j = 0$ and $d_j = 1$. Job j cannot begin until time 2 and by assumption completes at some time $f > 4$.

By Lemma 6, the total amount of time spent working on all jobs whose intervals are entirely contained within $[0, 4]$ is at most $4/2 = 2$ units of time. Moreover, there can be no gaps in the schedule during the interval $[2, f]$ because otherwise j would work during the gaps and finish earlier than time f . Finally, by the definition of THRASHING, there can be no work scheduled during $[2, f]$ on jobs having release dates before 0 because j has higher priority. Thus, f cannot be greater than 4 and we obtain a contradiction. \square

It may, of course, be unrealistically optimistic to give the online procrastinator the power to run arbitrarily fast. However, it follows from Theorem 7 that THRASHING never runs any job j faster than $4f_j(d_j)$. In fact, the proof of Theorem 7 indicates that we can reduce this upper bound still further to $2f_j(d_j)$ without increasing the max-stretch; we need only modify the speed functions so that the maximum job speed for job j is limited to $2f_j(d_j)$.

4 Conclusions

The first sentence of the conclusion, which summarizes the paper, is being written just a few hours before the deadline. As we were writing this paper, we were struck by the wealth of open problems in this area. For example, what is the right way to resolve the computational and numerical issues associated with linear and other speed functions? The scheduling problem becomes even more complex with speed functions that may be nonzero at jobs release times. For our online algorithm we did not try to optimize the constant in the online competitive ratio fully; what is the smallest that we can make this constant, especially where the speed functions are sub-linear?

We have also considered piecewise constant speed functions and have linear programming solutions for a number of different variants of the original problem. The constraints of the LP correspond to the constant speed intervals of each job and depend on the optimization metric. One important question is whether there are combinatorial algorithms that can be found for these formulations.

Finally, what about other metrics, especially in models where some jobs may be left unexecuted? What about settings where job streams are executed on parallel processors?

It is now several hours later, just minutes before the deadline. We were searching for the ideal way to end the paper and circumstances have unfortunately provided the answer. A campus-wide power failure at Stony Brook has cut two hours from our last-minute working time and highlights the difficulties of online scheduling for procrastinators.

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A boss had a quirky young worker
Whose performance was starting to irk her
But by procrastination
He met the occasion
Starting late and just working berserker.

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